Chaotic Behavior of the Double Pendulum

Experimental Physics

Laboratory Report 1
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Abstract

The double pendulum is an interesting example of a system which can undergo a chaotic motion. In this experiment, we attempt to determine whether the chaotic properties of the system are dependent on the release angle of the pendulum. We use rotational velocity data from an iPod which tracks the pendulum's motion in order to find the separation of the pendulum's angular position. If the separation grows exponentially, then the system is undergoing chaotic motion. We determine qualitatively whether the system is chaotic under specific initial conditions using the Lyapunov exponent. We find it to be positive in all three examined initial conditions: release angles 30°, 60° and 90°. Therefore our investigation concludes that within the angle range of our experiment, all initial conditions lead to chaotic motion of the double pendulum.

Introduction

Chaotic systems in nonlinear dynamics are those which depend sensitively on the initial conditions. This experiment explores the chaotic qualities of the double pendulum. Our goal is to determine whether there exists a point of release of the pendulum which does not result in chaotic behavior. Many natural systems exhibit chaotic behavior, for example weather and climate, population models in biology, or the orientation of Pluto's moons (NASA 2015).[1] Studying simple chaotic systems such as the double pendulum gives us a base to explore more complex ones which are found in nature.

The method to determine whether a system exhibits chaotic behavior is to find the Lyapunov exponent, λ, for $|\delta_n| = |\delta_0| \exp(n\lambda)$, where n is the number of iterations, δ_0 is the initial separation of the trajectories and δ_n is the separation of the trajectories at the nth iteration (Strogatz 1994).[2] If the system exhibits chaotic motion, $\lambda > 0$. The number n arises from the fact that the time considered is discrete. In our experiment, the number n corresponds to the nth measurement in a single data set.

The Liapunov exponent then is the tool that we can use to determine whether the double pendulum always exhibits the chaotic behavior or whether there exists an angle of release at which the chaotic behavior vanishes.

Method and Setup

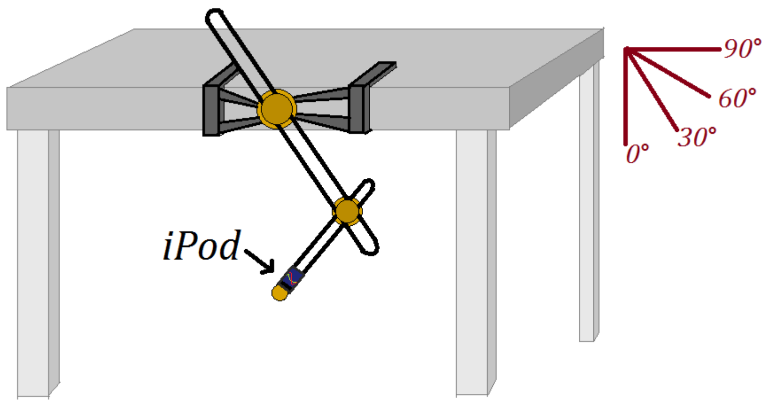


Figure 1: Our setup includes a double pendulum mounted to a table as showed on the drawing. The orientation of the release angles with respect to the pendulum is showed in red. The iPod's orientation with respect to the pendulum is showed further in fig. 2.

For the experiment, we use an Apple *iPod*'s built-in accelerometer. The data are collected through an application called *SensorLog*. It is set up to collect data 20 times a second. It collects proper acceleration and rotational velocity, both in 3 spacial dimensions (see fig. (2) for orientation of the axes). The maximal acceleration that the device can collect is $+2G$ and the minimal is $-2G$, where G is the Earth's gravitational acceleration, $9.81\,\mathrm{m/s^2}$.

The double pendulum is set up as shown in Figure (1). The pendulum is strapped to a heavy table using metal grip holders. The iPod is placed in a plastic bag and strapped to the outer edge of the double pendulum using gaffer tape.

The original goal was to graph the actual path of the iPod. In order to do this, we would have to combine the rotational velocity data with the acceleration data. However, early tests showed that the acceleration reached its maximum and minimum value quite easily, and so graphing the path of the pendulum became impossible. Therefore, under the assumption that one part of a chaotic system will approximately reflect the chaos of the whole system, we choose to focus solely on the data from the rotational velocity of the iPod.

To set up the 'drop position', i.e. the angle from where the pendulum is being set into motion, we use a release mechanism. The release mechanism consists of an elevation of which the height can be varied to which we attach a rigid metal rod.

The pendulum arm initially lays on the rod and is leveled to the desired angle. Once the angle is set, the rod is repeatedly slid from underneath the pendulum. The elevation height is not changed, which ensures that the angle of the release does not change significantly from one set of measurements to another. To find the drop angle as accurately as possible, we use a smart phone with the application '*Angle Meter Pro'*. We asses from the fluctuations in the application that the uncertainty on the angle is $2°$.

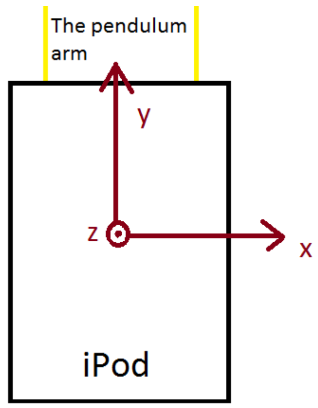


Figure 2: A scheme showing the orientation of the axes of the censor with respect to the iPod and the pendulum.

Item List

- Double Pendulum
- iPod Touch 2nd/3rd-Gen with built-in accelerometer
- Gaffer tape
- Plastic bag
- Rigid metal rod
- Phone based angle meter
- An elevation with changeable height

Results

We perform 10 measurements releasing from $(90 \pm 2)°$ angle, 10 measurements releasing from $(60 \pm 2)°$ angle and 10 measurements releasing from $(30 \pm 2)°$ angle. The angular distance traveled data from each angle are shown fig. 3), fig. 4) and fig. 5). Angular distance in this case is defined as how many radians the iPod has rotated in both positive and negative rotational directions from the starting position. Hence, all of the paths start at $0°$ away from the starting position (one of the three mentioned above) and proceed from there. Angular distance 2π radians means that the pendulum finished a full, anti-clockwise revolution from the starting angle.

We present the complete set of paths that we recorded for the initial angles $(90 \pm 2)°$ and $(60 \pm 2)°$ since the paths are easy to tell apart when graphed together. We refrain from that with the initial angle $(30 \pm 2)°$, as in this case all paths remain within the same range (the angle of release does not provide enough energy for the pendulum to make a full rotation). Therefore all paths are clumped together and are impossible to tell apart when plotted simultaneously. We instead present two examples of the comparison of two separate sets of data in fig. 5. We choose them such that one (left) presents the case where the paths remain close to each other throughout the full run of the experiment, while the other one (right) shows the case where the paths get separated further and further as time progresses. A further presentation of the spread of the data will be showed in the Analysis section (fig. 6c).

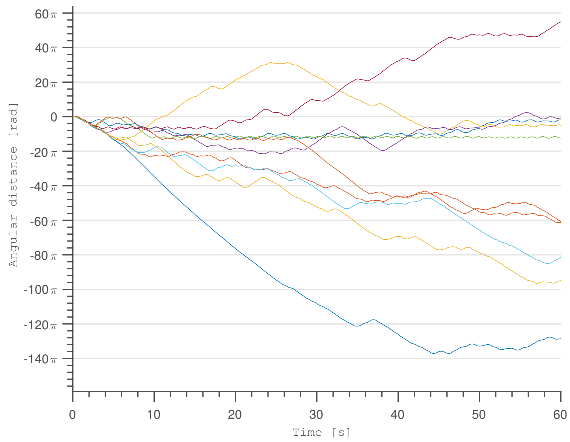


Figure 3: Angular distance data for 10 measurements releasing from the same position with a $(90 \pm 2)^{\circ}$ angle, an angular distance of 2π in this case means a full anti-clockwise(positive rotational direction) revolution.

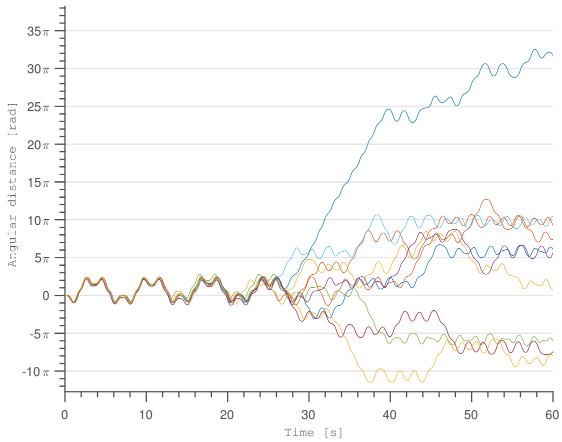


Figure 4: Angular distance data for 10 measurements releasing from the same position with a $(60 \pm 2)^{\circ}$ angle.

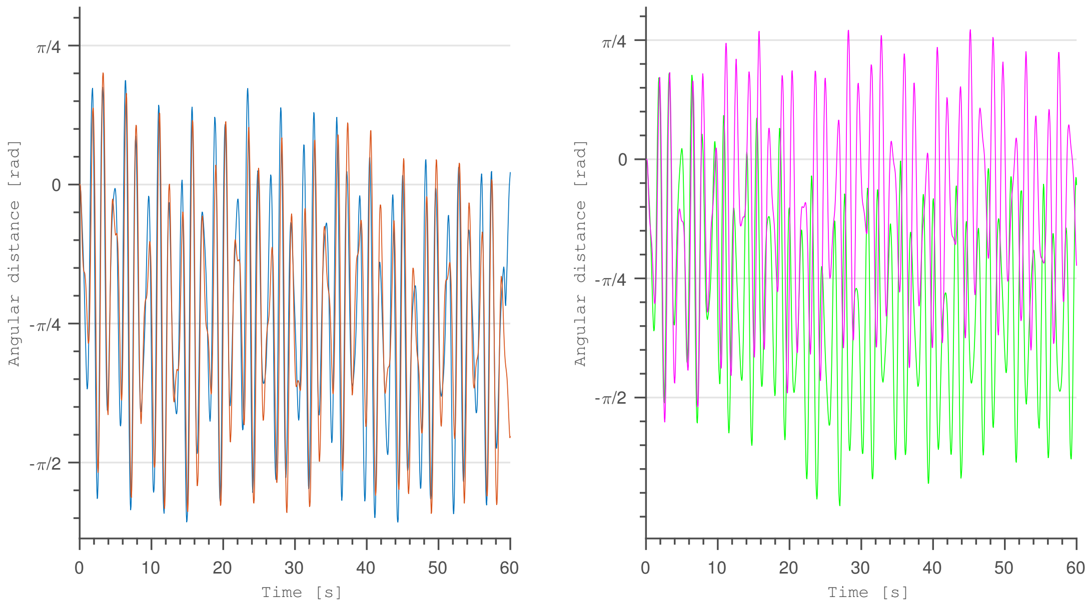


Figure 5: Angular distance data for 4 out of 10 measurements releasing from the same position with a $(30 \pm 2)^{\circ}$ angle. By comparing them 2 by 2 we can see that some outcomes are similar and some drastically different.

Analysis

In order to find the angular distance traveled by the iPod, we need to integrate the angular velocity data with respect to time. When plotting this, we notice a small shift in the angular distance, meaning that the iPod does not oscillate around the lowest point, In this case $-\pi/2$ if we release it at a 90° angle, but rather around $-\pi/2 + a$ where a converges to $\pi/2$ as time goes to infinity. This is due to the nature of the discrete data points and the low refresh rate around 20 Hz of the data.

It is possible to correct the shift with a linear angular distance compensation. By visualizing the angular distance traveled by the iPod, we are able to determine the shift at the end of the experiment and compensate for the angular shift (S) to get the correct angular distance. Because we assume it to be a linear shift, and because at $t = 0$, the angular shift is 0, we are able calculate the angular shift at a specific time, t using the linear formula

$$S = \frac{a}{t_{\text{end}}} \cdot t \tag{1}$$

where a is the total angular shift in the end of the experiment, t_{end} is the total time of the data set. The angular shift and the corrected angular distance data is visualized as shown in fig. 7a), fig. 7b) from the video[3] containing animations of the sensor data and the actual footage of one of the experiment measurements.

We can clearly see that the corrected data is a good representation of the orientation of the iPod, whereas the raw data starts out accurately, but eventually ends up shifted. This indicates that the shift is a main source of error. However, because the shift is within the range of $]0, \pi/2[$, less than a quarter of a revolution, and the same for all the measurements, we can still get comparable and valid data if we ignore the shift for all the measurements

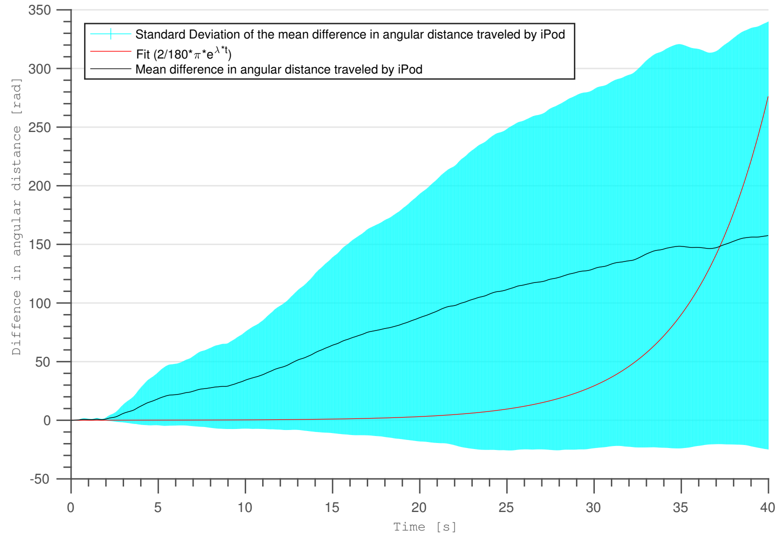
We now compare the paths of the angular distance traveled by the iPod for each of the starting conditions. Since we have 10 sets of data, we are able to make 45 comparisons for each of the data points. Hence for each time data point, we have 45 distinct values of the path separation δ, which we take the absolute value of. In order to find the Lyapunov exponent, we first find the mean of the δ values for each time data point. We then calculate the standard deviation for each $\bar{\delta}$ to illustrate the spread of the separation data. We then make an exponential Lyapunov fit through the mean of the separation data

$$|\delta(t)| = |\delta_0| \exp(t\lambda) \tag{2}$$

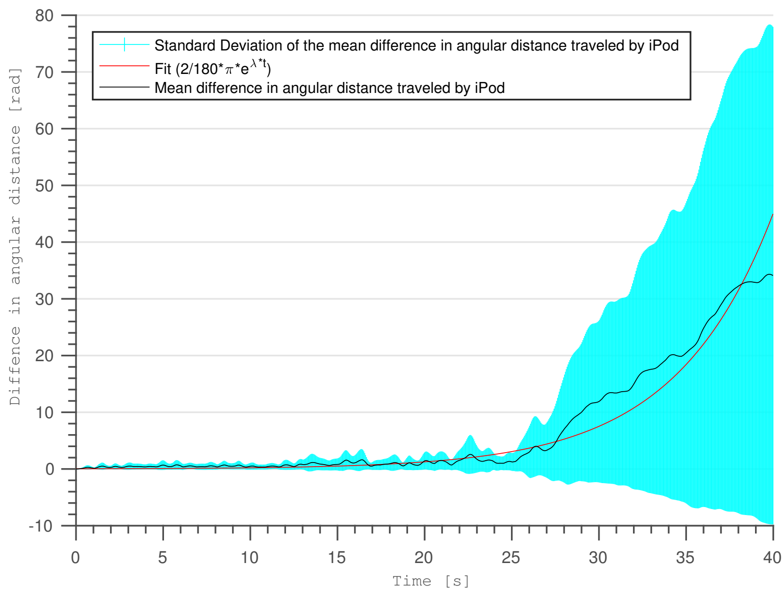
where $|\delta_0| = \frac{2}{180}\pi$ is the initial separation of the measurements from the same starting angle, in this case, it is the error of the starting angle in radians, since this is the maximum deviation of the initial condition we expect. $|\delta(t)|$ is the mean difference of angular distances of each measurement with the same starting angle at a specific time t.

The complete analyses are presented in fig. 6a), fig. 6b) and fig. 6c) with respective λ values $\lambda = 0.225 \pm 0.002$ for the $(90 \pm 2)^\circ$ starting angle, $\lambda = 0.179 \pm 0.002$ for the $(60 \pm 2)^\circ$ starting angle and $\lambda = 0.083 \pm 0.002$ for the $(30 \pm 2)^\circ$ starting angle.

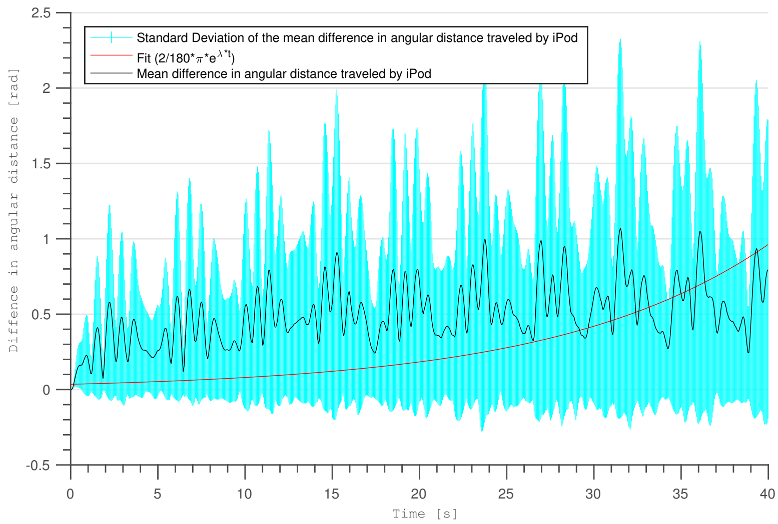
Positive Lyapunov exponents imply that the rotation of the secondary arm of the double pendulum is chaotic and hence implies that the motion of the double pendulum is chaotic.



(a) Mean difference in angular distance traveled by the iPod for 10 measurements releasing from the same position with a $(90 \pm 2)°$ angle (black), along with the standard deviation of difference in angular distance traveled by the iPod at each time point (blue) and the exponential fit (red).



(b) Mean difference in angular distance traveled by the iPod for 10 measurements releasing from the same position with a $(60 \pm 2)°$ angle (black), along with the standard deviation of difference in angular distance traveled by the iPod at each time point (blue) and the exponential fit (red).



(c) Mean difference in angular distance traveled by the iPod for 10 measurements releasing from the same position with a $(30 \pm 2)°$ angle (black), along with the standard deviation of difference in angular distance traveled by the iPod at each time point (blue) and the exponential fit (red).

Figure 6

When we consider how the different measurements take different paths, we notice that after a certain amount of time, the separation reaches a plateau. This was also predicted by Prof. Mogens Høgh Jensen when consulting the experiment with him. The separation in a chaotic system will grow exponentially, but as it is a closed system, it cannot grow to infinity. And as the pendulum tends to rest at a fixed position pointing straight downwards due to friction and gravity. Therefore, while analyzing the separation data, we choose to fit the exponential function only on the interval in which the plateau does not arise. This point is easy to choose in the case of the 60° release angle (see fig. 6b), however it is harder to see that point in the case of the two other release angles. Therefore we decide to use the $(60 \pm 2)°$ release angle as a reference and repeat the same procedure in the analysis of all release angles. With this in mind, we only evaluated the experiment from 0 seconds to 40 seconds, despite the experiment data continues to record for another 20 seconds for every measurement.

Discussion

The setup of our experiment has a number of significant limitations. The most important one that likely caused the largest error on our λ values, is the fact that we did not extract all of the information about the motion of the system, but a part of it - the rotational data of the secondary pendulum arm. Moreover, the double pendulum we used is built without the iPod. The extra weight of the iPod made the lower arm of the pendulum vibrate as it oscillated, which made parts of the double pendulum collide in a way that it lost energy. This happened the most at the $(90 \pm 2)°$ release angle and there were only a few collisions with the $(60 \pm 2)°$ release angle. We did not observe this behavior in the case of the $(30 \pm 2)°$ release angle.

Both of these limitations could be avoided if we used a different way of tracking the motion of the pendulum, for example recording a video of it oscillating and tracking the points on the pendulum that correspond to the motion of the two different arms. Alternatively, to avoid the collisions of the pendulum arms, a double pendulum could be build to meet the requirements of the experiment with a motion sensor included in both of the pendulum arms.

As can be seen on fig. 5 and 6c, our lowest release angle data show different patterns than the two other data sets. Instead of the paths growing further and further apart, we can observe a phase shift in an otherwise similar pattern of motion. The reason is that at this release point, the pendulum does not have enough energy to make a full rotation. Therefore all of the paths oscillate around the same point on the circle - this also affects our separation data and standard deviation. Since we only have the angular position data, it seems like the paths keep crossing each other, while in reality they are at two completely different points. This makes the analysis of the data sets from the $(30 \pm 2)^\circ$ release angle less valid. However, one can observe an increase in the separation of the paths on fig. 6c as time progresses, therefore it can be argued that it is still viable to seek a λ value for these data sets.

The standard deviations of fig. 6a), fig. 6b) and fig. 6c) appear too large to draw any conclusions from them. However, as standard deviation of the path difference δ describes the spread of the δ data points, it is a viable parameter to explore chaos with. Imagine a system with a standard deviation of the differences in angular distance infinitely close to 0. It would imply every path is at a equal distance to every other paths at a given time. It is impossible unless all the paths have a difference of 0 to each other which would mean the system results in the same out come every time. The system is hence by definition not chaotic. If we try to fit an exponential function through the mean difference of the data, it would fail, as the mean difference is 0 all the way along the x-axis. Therefore, on the contrary, because we have a large standard deviation in differences between each path, every path will have a different angular distance between each other which implies the paths deviate from each other. The system is therefore by definition chaotic and because we successfully fitted an exponential function through the data and it largely stayed within the range of standard deviation. Therefore we argue the Lyapounv exponent is valid for situations where $(90 \pm 2)^\circ$ and $(60 \pm 2)^\circ$ angle are the initial angular positions and it is to some extend valid for the $(30 \pm 2)^\circ$ initial angular position.

Conclusion

We set out to explore the chaotic nature of the double pendulum with the help from an accelerometer inside an iPod, hoping to recreate the path of the end of the double pendulum based on the acceleration data. However, due to the limitations of the accelerometer, we were only able to look at the rotational data of the iPod that is strapped onto the secondary arm of the double pendulum. The rotational data turns out to be able to partially reflect the chaotic motion of the double pendulum. After fitting the exponential function through the angular path separation δ for three different starting conditions, we got three different values for the Lyapunov exponent λ where

$\lambda = 0.225 \pm 0.002$ for the $(90 \pm 2)^\circ$ starting angle, $\lambda = 0.179 \pm 0.002$ for the $(60 \pm 2)^\circ$ starting angle and $\lambda = 0.083 \pm 0.002$ for the $(30 \pm 2)^\circ$ starting angle.

The fact that the Lyapunov exponents are positive indicates that the rotation of the secondary arm of the double pendulum is chaotic and hence implies the motion of the double pendulum is chaotic. However, we cannot conclude from our experimental data that there exists an angle of release for which the chaotic motion of the pendulum vanishes.

Acknowledgements

We would like to thank Prof. Mogens Høgh Jensen for a consultation in the chaos theory.

References

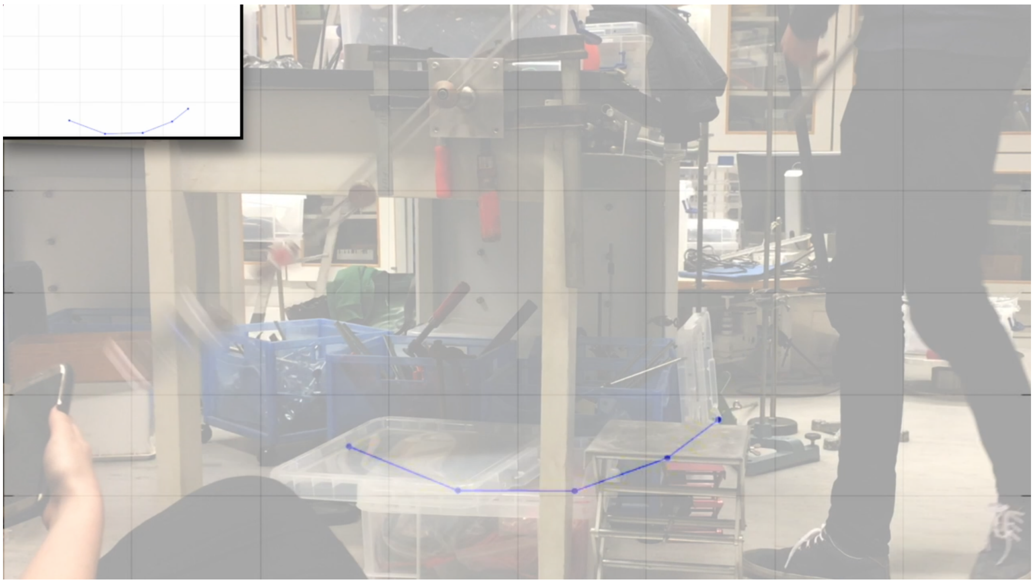
[1] Chou, Felicia, and Ray Villard. "NASA's Hubble Finds Pluto's Moons Tumbling in Absolute Chaos." *NASA*. Ed. Karen Northon. NASA, 03 June 2015. Web. 02 May 2017.

[2] Strogatz, Steven H. *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering.* N.p.: Addison-Wesley Publishing Company, 1994. Print.

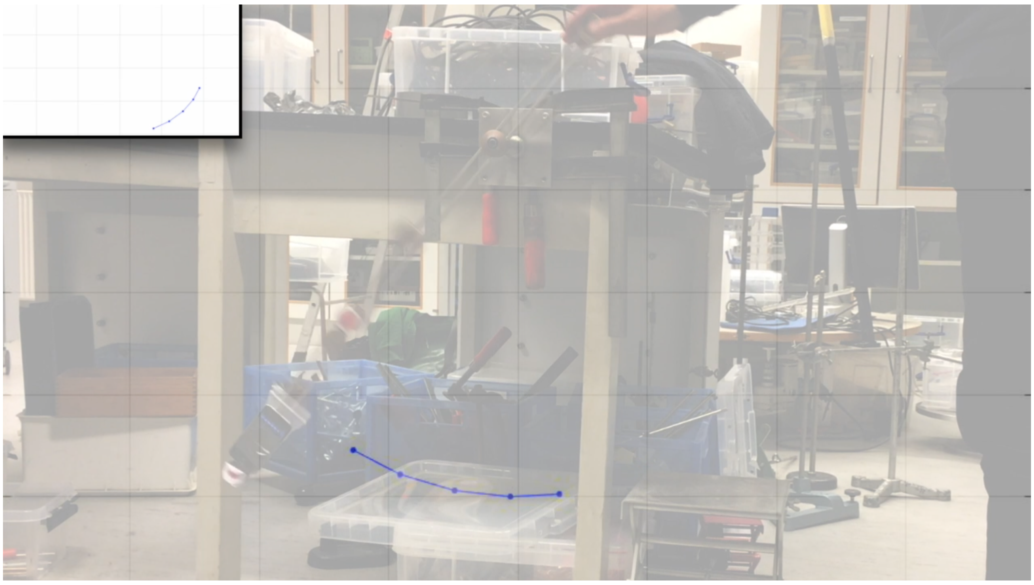
[3] Liu, Yifan. "Double Pendulum with iPod.", *YouTube*. YouTube, 30 April, 2017. Web. 2 May, 2017. `https://www.youtube.com/watch?v=hDfkjn6DqXg`

Appendix A

Stills from a video which shows the rotation of the pendulum in real time versus the plot of the rotational velocity of the iPod.



(a) At the beginning of the experiment. (Raw data on top left corner and corrected data with a correction of $a = -3/8\pi$ overlays on top of the actual footage of the experiment. NOTE: the data points is a real time representation of the orientation of the iPod, not the actual position of the iPod.) We can see that the raw data is in line with the corrected data



(b) Near the end of the experiment. We can see that raw data has shifted to the positive rotational direction while corrected data remains a good representation of the actual orientation of the iPod.