# Laplace Transform Intuition: What makes it so useful, and how you can "invent" it from scratch

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## 1 Introduction

The **Laplace Transform** $\mathcal{L}\{f(t)\}$ is an **integral transform** that is very useful for solving differential equations, but for many students (me included), it often feels mysterious and confusing. It's one thing to learn how to use it, but another to understand *why* it works the way it does. In this short article, I want to share the way I like to think about it by starting from what makes it so useful and working backwards from there to derive the Laplace Transform. We'll cover

1. What the Laplace Transform (and transforms in general) is,
2. What property the Laplace Transform has that makes it so useful for solving differential equations, and
3. How you could start from that property and *re-"discover"* the Laplace Transform for yourself, seeing why it works and how you could have invented it yourself like Laplace (or at least, this is how I like to imagine he found it).

## 2 Integral Transforms, and why the Laplace Transform is special

In general, an integral transform $T$ takes a function $f(t)$ and converts it into another function $F(s)$ in a new parameter $s$. For this discussion, we are specifically interested in integral transforms that integrate over the interval $[0, \infty)$:

$$T\{f(t)\} = F(s) = \int_0^\infty K(s,t) f(t) dt \tag{1}$$

The function $K(s,t)$ shown above is called the **kernel**. The Laplace Transform is one such integral transform defined with the kernel $K(s,t) = e^{-st}$:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

It can be shown through integration by parts that the Laplace Transform has the following property when integrating the derivative of a function $f(t)$:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{2}$$

Or more generally,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) \tag{3}$$

This property means that when we apply the Laplace transform to certain types of differential equations, it effectively converts them to **algebraic** ones that are much often easier to solve!

**Example 2.1.** Suppose we wish to solve the differential equation $f''(t)+3f'(t)+5f = 6$ with initial conditions $f(0) = 0$ and $f'(0) = 0$. Taking the Laplace transform using equation (3), we get:

$$\underbrace{(s^2 F(s) - sf(0) - f'(0))}_{\mathcal{L}\{f''(t)\}} + \underbrace{3(sF(s) - f(0))}_{\mathcal{L}\{3f'(t)\}} + \underbrace{5(F(s))}_{\mathcal{L}\{5f(t)\}} = \underbrace{\frac{6}{s}}_{\mathcal{L}\{6\}}$$

$$F(s)(s^2 + 3s + 5) + f(0)(-s - 3) - f'(0) = \frac{6}{s}$$

This is now an algebraic equation that can be solved for $F(s)$, and then for $f(s)$ via the inverse Laplace transform ($\mathcal{L}^{-1}$). A similar strategy works for solving other differential equations as well.

# 3 But...How would you ever figure that out? Derivation of the Laplace Transform

While many texts just introduce the Laplace transform and prove that it has the useful properties of converting differential equations into algebraic equations, I personally find it much more satisfying to explore how we can start from this goal of finding a transform that converts differential equations into algebraic ones and *derive* the Laplace transform from there.

**Goal** First, we state our goal. We seek to find some integral transform $T$ that, when applied to the derivative of a function ($f'(t)$), returns an algebraic expression in terms of that function ($f(t)$) and a variable $s$. In notation, this can be represented as

$$T\{f'(t)\} = p(s, f)T\{f(t)\} + q(s, f) \tag{4}$$

where $p(s,f)$ and $q(s,f)$ each represent some function of our new variable $s$ and the original function $f$ (we include both $s$ and $f$ as parameters to these functions because we *don't know yet* what they will look like - except that they won't contain a derivative of $f(t)$, since our entire goal here is to get rid of derivatives). $p(s,f)$ or $q(s,f)$ *might* look like $s^2 f(0)$, or it might look like something else. At this point, it doesn't really matter - we just need *some function* that this works for, since all we care about is getting this algebraic equation that we know how to solve.

**Note** that we *don't* provide $t$ as a parameter to these functions - so $p(s,f)$ and $q(s,f)$ would only be able to contain constant values of $f$ like $f(0)$ or $f(3)$, *not* $f(t)$.

If we find an integral transform that satisfies this property, then we will be able to convert differential equations into algebraic ones just as we did with the Laplace transform. Note (4)'s similarity to (2) - in (2), $p(s,f) = s$ and $q(s,f) = f(0)$. We will soon see this is what we get here later on.

**If at any point you feel lost in the derivation** that follows or forget what we're working towards, come back here and remind yourself what we're trying to achieve. We want a transform $T$ that acts according to (4) so that we can convert differential equations to algebraic ones. That's all that matters.

**Derivation** Remember what our integral transform $T$ looks like from equation (1), repeated here:

$$T\{f(t)\} = F(s) = \int_0^\infty K(s,t)f(t)dt$$

When we say we want to find an integral transform $T$, we really mean that we wish to determine the kernel $K$ that will give $T$ the properties we want.

So what is $K$? Well, let's start with (4) and mess with the left hand side (LHS) until we can get it to equal the right hand side (RHS). First, we'll expand out the $T$s into their full forms.

$$\underbrace{\int_0^\infty K(s,t)f'(t)dt}_{T\{f'(t)\}} = p(s,f)\underbrace{\int_0^\infty K(s,t)f(t)dt}_{T\{f(t)\}} + q(s,f)$$

We know we want the integral on the LHS to look like the one on the RHS, so let's integrate the LHS by parts ($u = K(s,t)$, $du = K'(s,t)dt$; $dv = f'(t)dt$, $v = f(t)$).

$$\underbrace{K(s,t)f(t)\Big|_0^\infty}_{uv} \underbrace{- \int_0^\infty K'(s,t)f(t)dt}_{-\int vdu} = p(s,f)\int_0^\infty K(s,t)f(t)dt + q(s,f) \quad (5)$$

First, we'll focus on the first term on the LHS. This notation is a little sloppy, since we can't *really* evaluate that expression at $\infty$, but it's okay if we take that notation to actually mean $\lim_{b\to\infty} K(s,t)f(t)\Big|_0^b$.

If we assume $K(s,t)f(t)$ converges to 0 when $t \to \infty$ (something we will make sure of later), then we get

$$K(s,t)f(t)\Big|_0^\infty = (0 - K(s,0)f(0)) = -K(s,0)f(0).$$

Because this term makes no reference to the variable $t$, we can call the entire thing $q(s,f)$... and great! Now that part, after re-arranging, matches the RHS of our main equation.

$$-\int_0^\infty K'(s,t)f(t)dt + q(s,f) = p(s,f)\int_0^\infty K(s,t)f(t)dt + q(s,f)$$

Now let's focus on the integral term in the LHS. We can see from inspection that it must equal the first term on the RHS, i.e.

$$\underbrace{-\int_0^\infty K'(s,t)f(t)dt}_{\text{from the LHS}} = \underbrace{p(s,f)\int_0^\infty K(s,t)f(t)dt}_{\text{from the RHS}} \tag{6}$$

How can we choose a kernel $K$ to make this work? Well, we just need to find a kernel such that

$$K'(s,t) = -p(s,f)K(s,t), \tag{7}$$

where $p$ is represents some function of $s$ and $f$ (just like on the RHS). *This assumption is extremely important, and will ultimately allow us to find K later.* If the assumption seems a little arbitrary or confusing, just imagine substituting it back into the LHS of (7). When we do that, we would be able to extract the $-p(s,f)$ term from the integral (since it doesn't rely on $t$ and is thus a "constant" during integration), giving us what we have on the RHS.

With this assumption in mind, substituting everything we've done back into the LHS, we get

$$\underbrace{p(s,f)\int_0^\infty K(s,t)f(t)dt + q(s,f) = p(s,f)\int_0^\infty K(s,t)f(t)dt + q(s,f)}_{\text{The LHS and RHS are identical now!}}$$

Awesome! Let's go over what just happened here. We started with equation (4), which specified that our transform must convert derivatives of a function into an algebraic expression involving a variable $s$ and the original function $f(t)$. Then, we expanded the transforms out and made two key assumptions that allowed us to show each side to be equal:

1. $K'(s,t) = -p(s,f)K(s,t)$
2. $\lim_{b\to\infty} K(s,t)f(t) = 0$

Our kernel $K$ *must* satisfy these assumptions for our transform to achieve our goal, so we can use them to figure out what $K$ must be. Let's start with assumption 1, which is a differential equation itself. Fortunately, we can solve it easily! (Note: I'm changing to Leibniz briefly to make the following steps more clear):

$$\frac{dK}{dt} = -p(s,f)K$$
$$\int \frac{dK}{K} = \int -p(s,f)dt$$
$$\ln|K| = -p(s,f)t$$

$$K = Ce^{-p(s,f)t}$$

Here we have a general form for our kernel $K$! To clean it up a bit, we can recall that our function $p(s,f)$ could be *anything* involving $s$ and $f$ - it could be $s^2$, $s + f(34)$, or something else; any of these should work. However, for the sake of simplicity (because remember, we want to use this kernel to help us convert differential equations to algebraic ones, which was the entire reason we went through all this trouble in the first place), we'll choose $p(s,f) = s$. $C$ can also be any constant, so let's choose 1 to make it nice.

With that we have our final kernel equation,

$$\boxed{K = e^{-st}} \tag{8}$$

So substituting back into our original definition for an integral transform (1), we get that our magical transform which will let us solve differential equations with ease is:

$$\boxed{T\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \mathcal{L}\{f(t)\}} \tag{9}$$

Which is, as it turns out, the **Laplace Transform**!

**Tying up loose ends** With our kernel in mind, we can ensure that assumption 2 is satisfied by mandating that all of our functions be of "exponential order", which just means that they don't grow faster than $e^{-st}$; i.e., there exist constants $M$, $c$, and $T$ such that

$$|f(t)| \leq Me^{ct} \tag{10}$$

for all values of $t \geq T$.