UNIVERSITY OF TEXAS AT AUSTIN

Methods of Applied Mathematics Problem Set 2

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February 1, 2015

1 Exercise 6.5

Let $1 \le p < \infty$ and suppose $f \in L^p(\mathbb{R})$. Let $g(x) = \int_x^{x+1} f(y)dy$. Prove that $g \in C_v(\mathbb{R})$.

Proof:
1. g is continuous: is to prove $\forall x \in \mathbb{R},\ \forall \epsilon > 0, \exists \delta, s.t\ \ |g(x+h) - g(x)| < \epsilon, \forall h < \delta$.
$\because$ continuous functions with compact support is dense in $L^p(\mathbb{R})$ space, $\therefore\ \exists\, \phi \in C_0(\mathbb{R}),\ s.t\ \|\phi - f\|_{L^p}$ is sufficient small, so that $\int_x^{x+1} |f - \phi| \le (\int_x^{x+1} |f - \phi|^p)^{1/p} (\int_x^{x+1} 1^q)^{1/q} \le \|f - \phi\|_{L^p} < \epsilon/3$.
$\because \phi$ is continuous with compact support, so it's actually uniformly compact. $\therefore \exists\delta$ s.t $|\phi(y+h) - \phi(y)| < \epsilon/3, \forall y \in \mathbb{R}$.

$$
\begin{aligned}
|g(x+h) - g(x)| &= |\int_{x+h}^{x+1+h} f(y)dy - \int_x^{x+1} f(y)dy| \\
&= |\int_x^{x+1} (f(y+h) - f(y))dy| \\
&\le \int_x^{x+1} |f(y+h) - \phi(y+h) + \phi(y+h) - \phi(y) + \phi(y) - f(y)|dy \\
&\le \int_x^{x+1} |f(y+h) - \phi(y+h)|dy + \int_x^{x+1} |\phi(y+h) - \phi(y)|dy + \int_x^{x+1} |\phi(y) - f(y)|dy \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\end{aligned}
$$

2. g vanishes in ∞
$\because f \in L^p(\mathbb{R}), \therefore \forall \epsilon > 0, \exists N \in \mathbb{R},\ s.t\ \ (\int_{|x|>N} |f(x)|^p dx)^{1/p} < \epsilon, \therefore$ when $x > max\{N, N+1\}, g(x) \le \int_x^{x+1} |f(x)|dx \le (\int_x^{x+1} |f(x)|^p dx)^{1/p} (\int_x^{x+1} 1^q)^{1/q} = (\int_x^{x+1} |f(x)|^p dx)^{1/p} \le \epsilon$.

2 EXERCISE 6.6

Show that the Fourier transform $\mathscr{F} : L^1(\mathbb{R}^d) \to C_v(\mathbb{R}^d)$ is not onto. Show, however, that its image is dense in $C_v(\mathbb{R}^d)$.
Proof:

3 EXERCISE 6.8

Give an example of a funciton $f \in L^2(\mathbb{R}^d)$ which is not in $L^1(\mathbb{R}^d)$, but such that $\hat{f} \in L^1(\mathbb{R}^d)$. Under what circumstances can this happen?
Solution:
$f(x) = \frac{\sin x}{x}$, $\hat{f} = 1/2\sqrt{\pi/2}(\text{Sign}(1-\xi) + \text{Sign}(\xi+1))$, here f is in L^2/L^1 but $\hat{f}$ is in L^1.

4 EXERCISE 6.29

Prove that if $\{f_j\}_{j=1}^{\infty} \subset S$ and $f_j \xrightarrow{S} f$, then for any $1 \le p \le \infty$, $f_j \xrightarrow{L^p} f$.
Proof:

$$\begin{aligned}
\because f_j \xrightarrow{S} f, \quad \therefore \quad & \rho_n(f_j - f) \to 0, \forall n \in \mathbb{N}, \\
i.e \quad & \|(1+|\cdot|^2)^{n/2} D^\alpha (f_j - f)\|_{L^\infty} \to 0, \forall n \in \mathbb{N}, \forall |\alpha| \le n, \alpha \in \mathbb{N}^d. \\
\text{Meanwhile, } \|f_j - f\|_{L^p}^p \quad = \quad & \int |f_j(x) - f(x)|^p dx \\
= \quad & \int_{B_1(0)} |f_j(x) - f(x)|^p dx + \int_{|x|\ge 1} |f_j(x) - f(x)|^p dx
\end{aligned}$$

The former integral obviously can be restrained to any small value, so consider the latter. Since for any small ϵ, $\exists N$, for $\forall j \ge N$, $(1+|x|^2)^{(d+1)/2}(f_j - f) < \epsilon$. Then

$$\begin{aligned}
\int_{|x|\ge 1} |f_j(x) - f(x)|^p dx &= \int_{|x|\ge 1} |x|^{-p(d+1)} |x|^{p(d+1)} |f_j(x) - f(x)|^p dx \\
&< \int_{|x|\ge 1} |x|^{-p(d+1)} ((1+|x|^2)^{(d+1)/2} |f_j(x) - f(x)|)^p dx \\
&\le \epsilon^p \int_{|x|\ge 1} |x|^{-p(d+1)} dx \\
&= \epsilon^p d\omega_d \int_1^\infty r^{-p(d+1)} r^{d-1} dr \\
&< \epsilon^p d\omega_d \int_1^\infty r^{-2} dr \\
&= \epsilon^p
\end{aligned}$$

where $d\omega_d$ is the measure of the unit sphere. So all in all $f_j \xrightarrow{L^p} f$.

5 EXERCISE 6.31

Let $f \in H^s(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : (1+|\xi|^2)^{s/2}|\hat{f}(\xi)| \in L^2(\mathbb{R}^d)\}$.
(a) Show that there is some $s_0 \in \mathbb{R}$ such that $\hat{f}(\xi) \in L^1(\mathbb{R}^d)$ for $s > s_0$.
(b) Apply the Riemann-Lebesgue Lemma to $\hat{f}(\xi)$ to show that, for $s > s_0$, there is some continuous function g such that $f = g$ almost everywhere.
Solution:
(a) Suppose $s_0 = d+1$. When $s \geq s_0$, take $g(\xi) = (1+|\xi|^2)^{s/2}|\hat{f}(\xi)|$, so $g \in L^2(\mathbb{R}^d)$. Apparently $\hat{f}(\xi) = \frac{g}{(1+|\xi|^2)^{s/2}}$, and $|\hat{f}| \leq |g|$

$$\begin{aligned}\int_{\mathbb{R}^d} |\hat{f}(x)|dx &= \int_{B_1(0)} |\hat{f}(x)|dx + \int_{|x|\geq 1, |g(x)|\leq 1} |\hat{f}(x)|dx + \int_{|x|\geq 1, |g(x)|>1} |\hat{f}(x)|dx \\ &\leq \int_{B_1(0)} |g(x)|dx + \int_{|x|\geq 1, |g(x)|\leq 1} \frac{1}{(1+|x|^2)^{s/2}} + \int_{|x|\geq 1, |g(x)|>1} |g(x)|^2 dx \quad (5.1)\end{aligned}$$

For the first term of (5.1), we know in bounded area $L^1 \subset L^2$, so $g|_{B_1(0)} \in L^1(B_1(0))$. The first term$< \infty$.
For the second term of (5.1), we know that $s \geq d+1$, so it $< \int_{|x|\geq 1, |g(x)|\leq 1} |x|^{-d-1} \leq \int_{|x|\geq 1} |x|^{-d-1} = d\omega_d \int_1^\infty r^{-d-1} r^{d-1} dr < \infty$.
For the third term of (5.1), as $g \in L^2$, it obviously $< \infty$.

(b)