On Finding Peak Height of Projectile Using Newtonian Mechanics and Time Measuring Device

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Introduction

One can start by reviewing simple definitions of velocity and acceleration as shown below,

$$v \equiv \lim_{\Delta t \to 0} \left[\frac{\Delta x}{\Delta t} \right]$$

where v is velocity, Δx is the displacement, and Δt is the change in time, and

$$a \equiv \lim_{\Delta t \to 0} \left[\frac{\Delta v}{\Delta t} \right]$$

where Δv represents change in velocity. The equations for average velocity and acceleration can be written by changing infinitesimal changes from the previous equations for these quantities into differences of initial and final conditions as such,

$$\overline{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{x_2 - x_1}{t}$$

$$\overline{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{v_2 - v_1}{t}$$

These quantities are also defined as

$$\overline{v} = \frac{v_2 - v_1}{2}$$

$$\overline{a} = \frac{a_2 - a_1}{2}$$

Considering that displacement is the difference between final and initial positions, one can formulate it as such,

$$d = x_2 - x_1$$

Substituting that into the previous equation for average velocity, one is left with

$$\overline{v} = \frac{d}{t}$$

and by multiplying both sides by t the formula for displacement emerges,

$$d = \overline{v}t$$

If one multiplies acceleration given by

$$a = \frac{v_2 - v_1}{t}$$

by displacement given by

$$d = \overline{v}t$$

one is left with

$$ad = \frac{v_2 - v_1}{t}\frac{v_1 + v_2}{2}t$$
$$ad = \frac{(v_2 - v_1)(v_2 + v_1)}{2}$$
$$ad = \frac{v_2^2 - v_1^2}{2}$$
$$v_2^2 - v_1^2 = 2ad$$

These are all of the formulas needed to solve the problem.

The Problem

The problem's purpose is to essentially find the peak height of a launched projectile only knowing the time of flight, presumably using a device in likeness to a stopwatch. The initial velocity is unknown. The following equations can be referred to,

$$v_2^2 - v_1^2 = 2ad$$
$$a = \frac{v_2 - v_1}{t}$$

By rearranging the equation for acceleration as follows, one can express v_1 in terms of acceleration and final velocity.

$$a = \frac{v_2 - v_1}{t}$$

Multiply both sides by t,

$$at = v_2 - v_1$$

Subtract v_2 from both sides,

$$at - v_2 = -v_1$$

Multiply both sides by -1,

$$v_1 = v_2 - at$$

Plug this back into the equation to have

$$v_2^2 - (v_2 - at)^2 = 2ad$$

For now, this is being solved for only half of the parabola and for vertical velocity, and therefore the final velocity at the peak will be zero, making this equation

$$-(-at)^2 = 2ad$$

$$-a^2t^2 = 2ad$$

Because the problem deals with a as a gravitational force equivalent, one can say that $a = -g$, and substituting this into the equation gives

$$-g^2t^2 = -2gd$$

Divide both sides by $-g$,

$$gt^2 = 2d$$

The time is only half of the total time, so one can express t as half of the total time T,

$$g\left(\frac{1}{2}T\right)^2 = 2d$$

$$\frac{gT^2}{4} = 2d$$

Divide both sides by 2,

$$\frac{gT^2}{8} = d$$

In general, g can stand for any given gravitational force equivalent, yet it is most commonly used for Earth's gravitational acceleration at about sea level. And that is how to find the maximum height d of a projectile only knowing the total time.