Modeling Population Dynamics

Raul Guajardo

University of Texas - Pan-American

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Introduction

- Population dynamics can be represented as Ordinary Differential Equations, although it is not exactly precise, it can be extremely useful in dealing with problems like population control, medical treatment etc.
- For the following slides we will let `p(t)` represent the total number of members of a population at time `t`.
- Below we first look at a steady production model.

Steady Production

If population increase is determined by a constant. Our ODE model is very simple and shown below.

$$\frac{dp}{dt} = c; p(0) = p_0 \Rightarrow p(t) = ct + p_0.$$

Malthusian model

- The malthusian model assumes that the `birth rate` and `death rate` of a population are proportional to the current size of the population.
- To include it in your document, use the `includegraphics` command (see the comment below in the source code).

Malthusian Model

In this case we will let `b` represent birth rate and `d` represent death rate.

$$\frac{dp}{dt} = bp - dp = rp; p(0) = p_0 \Rightarrow p(t) = p_0 exp(rt).$$

Where r will be referred to as the `growth parameter` of the population.

Logistic Model

- A problem with the previous model is that it assumes that there is no maximum limit for the size of the population.
- In reality that is not true, the environment can't sustain a population that is always increasing.
- Therefore we introduce `K`, the `"carrying capacity"`.
- In other words `K` is the greatest population sustainable by the environment.

Logistic Model

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}); p(0) = p_0 \Rightarrow p(t) = \frac{p_0 K}{(K - p_0)exp(-rt) + p_0}.$$

Logistic Model Analysis

We will analyze the logistic ODE model and determine the functionality of its critical points. In order to solve for the critical points we must set

$$\frac{dp}{dt} = 0$$

In this case we are left with the equation...

$$rp_e(1 - \frac{p_e}{K}) = 0 \Rightarrow p_e = 0, K.$$

Where 0 and K are the equilibrium points of the logistic model.

Logistic Model Analysis

We can see how the critical points behave by checking the sign of $\frac{dp}{dt}$ on either side of the critical points.

- For $p_e = K$
- We can see that at $p_e > K$, $\frac{dp}{dt} < 0$. In other words when the population is greater than `K`, the `carrying capacity`, we see that the population decreases.
- As for when $p_e < K$, $\frac{dp}{dt} > 0$. Which means that when the population is less than K , the population increases towards K.
- Therefore K is a stable critical point, since the population always approaches K from either side.

Logistic Model Analysis

- For $p_e = 0$
- We can see that at $p_e > 0$, $\frac{dp}{dt} > 0$. In other words when the population is greater than 0, we see that the population will increase away from the 0.
- As for $p_e < 0$, we ignore it because we cannot have a negative population.
- Therefore 0 is an unstable critical point, since the population can only increase from 0.

Resources

Modeling with ODE
P. Howard
Fall 2009
<http://www.math.tamu.edu/~phoward/m442/mododе.pdf>

Any Questions?