Riemann's Rearrangement Theorem

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Abstact

"(Infinite) series are the invention of the devil, by using them, on may draw any conclusion he pleases, and that is why these series have produced so many fallacies and so many paradoxes."
-Neils Hendrik Abel

As an example of what can go wrong, we will look at the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ and set this equal to S.

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \ldots \quad (1)$$

Multiply both sides by 2:

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \ldots \quad (2)$$

Collect terms with the same denominator and simplify:

$$2S = (2 - 1) - \frac{1}{2} + (\frac{2}{3} - \frac{1}{3}) - \frac{1}{4} + (\frac{2}{5} - \frac{1}{5}) - \frac{1}{6} + \ldots \quad (3)$$

We arrive at this:

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \quad (4)$$

We see that on the right side of this equation we have the same series we started with. In other words, by combining equations 1 and 4, we obtain:

$$2S = S. \tag{5}$$

We then divide by S, and have shown that:

$$2 = 1. \tag{6}$$

Peter Lejeune-Dirichlet

Figure: Peter Lejeune-Dirichlet

Berhard Riemann

Figure: Georg Friedrich Bernhard Riemann

Theorem: Part One

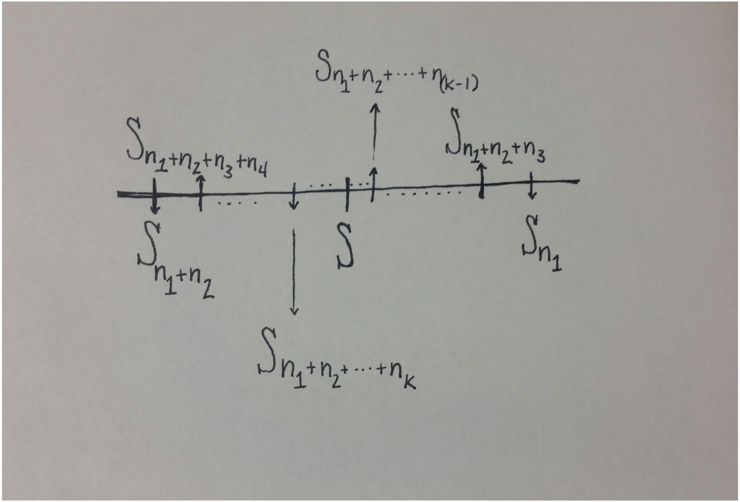
In a conditionally convergent series, the sum of the positive terms is a divergent series and the sum of the negative terms is a divergent series.

$\sum a_n$ is a conditionally convergent series
Let $\sum a_n^-$ represent the negative terms of $\sum a_n$
Let $\sum a_n^+$ represent the positive terms of $\sum a_n$
Then $\sum a_n = \sum a_n^+ + \sum a_n^-$

Case 1: $\sum a_n^+$ and $\sum a_n^-$ **both converge**. (NOT POSSIBLE)
Case 2: $\sum a_n^+$ **converges** and $\sum a_n^-$ **diverges** (NOT POSSIBLE)
Case 3: $\sum a_n^+$ **diverges** and $\sum a_n^-$ **converges** (NOT POSSIBLE)
Case 4: $\sum a_n^+$ and $\sum a_n^-$ **both diverge** (Satisfies Theorem)

Theorem: Part Two

Let $\sum a_n$ be a conditionally convergent series, and let S be a given real number. Then a rearrangement of the terms of $\sum a_n$ exists that converges to S.


$S_{n_1+n_2+\cdots+n_{(k-1)}}$
$S_{n_1+n_2+n_3+n_4}$
$S_{n_1+n_2+n_3}$
$S_{n_1+n_2}$
S
S_{n_1}
$S_{n_1+n_2+\cdots+n_k}$

$$S_{n_1+n_2+n_3+\ldots+n_k} \to S$$

Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that if $k \geq N$, then

$$\mid S_{n_1+n_2+n_3+\ldots+n_k} - S \mid < \epsilon.$$

Then, for $n_k \geq N_1$ we have,

$$\mid S_{n_1+n_2+n_3+\ldots+n_k} - S \mid$$

$$\leq \mid S_{n_1+n_2+n_3+\ldots+n_{k-1}} - S_{n_1+n_2+n_3+\ldots+n_k} \mid$$

$$= \mid a_{n_k} \mid$$

$$< \epsilon$$

Therefore, we have found a $k \geq N$ such that

$$\mid S_{n_1+n_2+n_3+\ldots+n_k} - S \mid < \epsilon$$

.

insert code and explain why (picture)

References

Galanor, Stewart
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Mathematics Teacher Vol.(80) 675 – 681.

Riemann, Bernard
Uber die Darstellbarkeit einer Function durch eine triogonometrische Reihe
Gesammelte Mathematische Werke (Leipzig 1876): 213-53

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